Spectral analysis of Markov kernels and application to the convergence rate of discrete random walks

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Abstract

Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain on a measurable space $\mathbb{X}$ with transition kernel P and let $V : \mathbb{X} \to [1,+\infty)$. The Markov kernel P is here considered as a linear bounded operator on the weighted-supremum space $\mathcal{B}_V$ associated with V. Then the combination of quasi-compactness arguments with precise analysis of eigen-elements of P allows us to estimate the geometric rate of convergence $\rho_V(P)$ of $\{X_n\}_{n\in\mathbb{N}}$ to its invariant probability measure in operator norm on $\mathcal{B}_V$. A general procedure to compute $\rho_V(P)$ for discrete Markov random walks with identically distributed bounded increments is specified.

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1 Introduction

Let $(\mathbb{X},\mathcal{X})$ be a measurable space with a σ-field $\mathcal{X}$, and let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space $\mathbb{X}$ and transition kernels $\{P(x,\cdot) : x \in \mathbb{X}\}$. Let $V : \mathbb{X} \to [1,+\infty)$. Assume that $\{X_n\}_{n\geq 0}$ has an invariant probability measure π such that $\pi(V) := \int_{\mathbb{X}} V(x)\pi(dx) < \infty$. This paper is based on the connection between spectral properties of the Markov kernel P and the so-called V-geometric ergodicity [MT93] which is the following convergence property for some constants $c_\rho > 0$ and $\rho \in (0,1)$:

$$\sup_{|f|\leq V}\sup_{x\in\mathbb{X}} \frac{\big|\mathbb{E}[f(X_n) \mid X_0 = x] - \pi(f)\big|}{V(x)} \leq c_\rho\, \rho^n. \tag{1}$$

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Let us introduce the weighted-supremum Banach space $(\mathcal{B}_V, \|\cdot\|_V)$ composed of measurable functions $f : \mathbb{X} \to \mathbb{C}$ such that

$$\|f\|_V := \sup_{x\in\mathbb{X}} \frac{|f(x)|}{V(x)} < \infty.$$

Then (1) reads as $\|P^n f - \pi(f)1_{\mathbb{X}}\|_V \leq c_\rho \rho^n$ for any $f \in \mathcal{B}_V$ such that $\|f\|_V \leq 1$, and there is a great interest in obtaining upper bounds for *the convergence rate* $\rho_V(P)$ defined by

$$\rho_V(P) := \inf\big\{\rho \in (0,1), \sup_{\|f\|_V\leq 1} \|P^n f - \pi(f)1_{\mathbb{X}}\|_V = O(\rho^n)\big\}. \tag{2}$$

For irreducible and aperiodic discrete Markov chains, criteria for the V-geometric ergodicity are well-known from the literature using either the equivalence between geometric ergodicity and V-geometric ergodicity of $\mathbb{N}$-valued Markov chains [HS92, Prop. 2.4], or the strong drift condition. For instance, when $\mathbb{X} := \mathbb{N}$ (with $\lim_n V(n) = +\infty$), the strong drift condition is

$$PV \leq \varrho V + b\, 1_{\{0,1,\dots,n_0\}}$$

for some $\varrho < 1, b < \infty$ and $n_0 \in \mathbb{N}$ (see [MT93]). Estimating $\rho_V(P)$ from the parameters ϱ, b, n_0 is a difficult issue. This often leads to unsatisfactory bounds, except for stochastically monotone P (see [MT94, LT96, Bax05] and the references therein).

This work presents a new procedure to study the convergence rate $\rho_V(P)$ under the following weak drift condition

$$\exists N \in \mathbb{N}^*,\ \exists d \in (0,+\infty),\ \exists \delta \in (0,1), \quad P^N V \leq \delta^N\, V + d\, 1_{\mathbb{X}}. \tag{\textbf{WD}}$$

The V-geometric ergodicity clearly implies (**WD**). Conversely, such a condition with $N = 1$ was introduced in [MT93, Lem. 15.2.8] as an alternative to the drift condition [MT93, (V4)] to obtain the V-geometric ergodicity under suitable assumption on V. Note that, under Condition (**WD**), the following real number $\delta_V(P)$ is well defined:

$$\delta_V(P) := \inf\big\{\delta \in [0,1) : \exists N \in \mathbb{N}^*, \exists d \in (0,+\infty),\ P^N V \leq \delta^N\, V + d\, 1_{\mathbb{X}}\big\}.$$

A spectral analysis of P is presented in Section 2 using quasi-compactness. More specifically, when the Markov kernel P has an invariant probability distribution, the connection between the V-geometric ergodicity and the quasi-compactness of P is made explicit in Proposition 2.1. Namely, P is V-geometrically ergodic if and only if P is a power-bounded quasi-compact operator on $\mathcal{B}_V$ for which $\lambda = 1$ is a simple eigenvalue and the unique eigenvalue of modulus one. In this case, if $r_{ess}(P)$ denotes the essential spectral radius of P on $\mathcal{B}_V$ (see (5)) and if $\mathcal{V}$ denotes the set of eigenvalues λ of P such that $r_{ess}(P) < |\lambda| < 1$, then the convergence rate $\rho_V(P)$ is given by (Proposition 2.1):

$$\rho_V(P) = r_{ess}(P) \ \text{ if }\ \mathcal{V} = \emptyset \quad \text{ and } \quad \rho_V(P) = \max\{|\lambda|, \lambda \in \mathcal{V}\} \ \text{ if }\ \mathcal{V} \neq \emptyset. \tag{3}$$

Interesting bounds for generalized eigenfunctions $f \in \mathcal{B}_V \cap \operatorname{Ker}(P - \lambda I)^p$ associated with $\lambda \in \mathcal{V}$ are presented in Proposition 2.2. Property (3) is relevant to study the convergence rate $\rho_V(P)$ provided that, first an accurate bound of $r_{ess}(P)$ is known, second the above set $\mathcal{V}$ is available. Bounds of $r_{ess}(P)$ related to drift conditions can be found in [Wu04] and [HL14] under various assumptions (see Subsection 2.1). In view of our applications, let us just mention that $r_{ess}(P) = \delta_V(P)$ in case $\mathbb{X} := \mathbb{N}$ and $\lim_n V(n) = +\infty$ (see Proposition 3.1). However, even if the state space is discrete, finding the above set $\mathcal{V}$ is difficult.

In Section 3, the above spectral analysis is applied to compute the rate of convergence $\rho_V(P)$ of discrete Random Walks (RW). In particular, a complete solution is presented for RWs with identically distributed (i.d.) bounded increments. In fact, Proposition 3.4 allows us to formulate an algebraic procedure based on polynomial eliminations providing $\rho_V(P)$ (see Corollary 4.1). To the best of our knowledge, this general result is new. Note that it requires neither reversibility nor stochastic monotonicity of P.

This procedure is illustrated in Section 4. First we consider the case of birth-and-death Markov kernel P defined by $P(0,0) := a$ and $P(0,1) := 1-a$ for some $a \in (0,1)$ and by

$$\forall n \geq 1,\ P(n,n-1) := p, \quad P(n,n) := r, \quad P(n,n+1) := q,$$

where $p,q,r \in [0,1]$ are such that $p+r+q = 1$, $p > q > 0$. Explicit formula for $\rho_V(P)$ with respect to $V := \{(p/q)^{n/2}\}_{n\in\mathbb{N}}$ is given in Proposition 4.1. When $r := 0$, such a result has been obtained for $a < p$ in [RT99] and [Bax05, Ex. 8.4] using Kendall's theorem, and for $a \geq p$ in [LT96] using the stochastic monotony of P. Our method gives a unified and simpler computation of $\rho_V(P)$ which moreover encompasses the case $r \neq 0$. For general RWs with i.d. bounded increments, the elimination procedure requires to use symbolic computations. The second example illustrates this point with the non reversible RW defined by

$$\forall n \geq 2,\ P(n,n-2) = a_{-2},\ P(n,n-1) = a_{-1},\ P(n,n) = a_0,\ P(n,n+1) = a_1$$

for any nonnegative a_i satisfying $a_{-2} + a_{-1} + a_0 + a_1 = 1$, $a_{-2} > 0$, $2a_{-2} + a_{-1} > a_1 > 0$, and for any finitely many boundary transition probabilities. In Section 5, specific examples of RWs on $\mathbb{X} := \mathbb{N}$ with unbounded increments considered in the literature are investigated.

To conclude this introduction, we mention a point which can be source of confusion in a first reading. In this paper, we are concerned with the convergence rate (2) with respect to some weighted-supremun Banach space $\mathcal{B}_V$. Thus, we do not consider here the decay parameter or the convergence rate of ergodic Markov chains in the usual Hilbert space $\mathbb{L}^2(\pi)$ which is related to spectral properties of the transition kernel with respect to this space. In particular, for Birth-and-Death Markov chains, we can not compare our results with those of [vDS95] on the $\ell^2(\pi)$-spectral gap and the decay parameter. A detailed discussion is provided in Remark 4.2.

2 Quasi-compactness on $\mathcal{B}_V$ and V-geometric ergodicity

We assume that P satisfies (**WD**). Then P continuously acts on $\mathcal{B}_V$, and iterating (**WD**) shows that P is power-bounded on $\mathcal{B}_V$, namely $\sup_{n\geq 1} \|P^n\|_V < \infty$, where $\|\cdot\|_V$ also stands for the operator norm on $\mathcal{B}_V$. Thus we have $r(P) := \lim_n \|P^n\|_V^{1/n} = 1$ since P is Markov.

2.1 From quasi-compactness on $\mathcal{B}_V$ to V-geometric ergodicity

Let I denote the identity operator on $\mathcal{B}_V$. Recall that P is said to be quasi-compact on $\mathcal{B}_V$ if there exist $r_0 \in (0,1)$ and $m \in \mathbb{N}^*$, $\lambda_i \in \mathbb{C}$, $p_i \in \mathbb{N}^*$ $(i = 1, \ldots, m)$ such that:

$$\mathcal{B}_V = \overset{m}{\underset{i=1}{\oplus}} \operatorname{Ker}(P - \lambda_i I)^{p_i} \oplus H, \tag{4a}$$

where the λ_i's are such that

$$|\lambda_i| \geq r_0 \quad \text{and} \quad 1 \leq \dim \operatorname{Ker}(P - \lambda_i I)^{p_i} < \infty, \tag{4b}$$

and H is a closed P-invariant subspace such that

$$\inf_{n \geq 1} \big(\sup_{h \in H,\, \|h\| \leq 1} \|P^n h\| \big)^{1/n} < r_0. \tag{4c}$$

Concerning the essential spectral radius of P, denoted by $r_{ess}(P)$, here it is enough to have in mind that, if P is quasi-compact on $\mathcal{B}_V$, then we have (see for instance [Hen93])

$$r_{ess}(P) := \inf \big\{ r_0 \in (0,1) \text{ such that (4a)-(4c) hold} \big\}. \tag{5}$$

As mentioned in Introduction, the essential spectral radius of Markov kernels acting on $\mathcal{B}_V$ is studied in [Wu04, HL14]. For instance, under Condition (**WD**), the following result is proved in [HL14]: if P^ℓ is compact from $\mathcal{B}_0$ to $\mathcal{B}_V$ for some $\ell \geq 1$, where $(\mathcal{B}_0, \|\cdot\|_0)$ is the Banach space composed of bounded measurable functions $f : \mathbb{X} \to \mathbb{C}$ equipped with the supremum norm $\|f\|_0 := \sup_{x \in \mathbb{X}} |f(x)|$, then P is quasi-compact on $\mathcal{B}_V$ with

$$r_{ess}(P) \leq \delta_V(P).$$

Moreover, equality $r_{ess}(P) = \delta_V(P)$ holds in many situations, in particular in the discrete state case with $V(n) \to \infty$ (see Proposition 3.1).

Next we explicit a result which makes explicit the relationship between the quasi-compactness of P and the V-geometric ergodicity of the Markov chain $\{X_n\}_{n \in \mathbb{N}}$ with transition kernel P. Moreover, we provide an explicit formula for $\rho_V(P)$ in terms of the spectral elements of P. Note that for any $r_0 \in (r_{ess}(P), 1)$, the set of all the eigenvalues of λ of P such that $r_0 \leq |\lambda| \leq 1$ is finite (use (5)).

Proposition 2.1 *Let P be a transition kernel which has an invariant probability measure π such that $\pi(V) < \infty$. The two following assertions are equivalent:*

(a) P is V-geometrically ergodic.

(b) P is a power-bounded quasi-compact operator on $\mathcal{B}_V$, for which $\lambda = 1$ is a simple eigenvalue (i.e. $\operatorname{Ker}(P - I) = \mathbb{C} \cdot 1_{\mathbb{X}}$) and the unique eigenvalue of modulus one.

Under any of these conditions, we have $\rho_V(P) \geq r_{ess}(P)$. In fact, for $r_0 \in (r_{ess}(P), 1)$, denoting the set of all the eigenvalues λ of P such that $r_0 \leq |\lambda| < 1$ by $\mathcal{V}_{r_0}$, we have:

- *either* $\rho_V(P) \leq r_0$ *when* $\mathcal{V}_{r_0} = \emptyset$,
- *or* $\rho_V(P) = \max\{|\lambda|,\, \lambda \in \mathcal{V}_{r_0}\}$ *when* $\mathcal{V}_{r_0} \neq \emptyset$.

Moreover, if $\mathcal{V}_{r_0} = \emptyset$ *for all* $r_0 \in (r_{ess}(P), 1)$*, then* $\rho_V(P) = r_{ess}(P)$*.*

The V-geometric ergodicity of P obviously implies that P is quasi-compact on $\mathcal{B}_V$ with $\rho_V(P) \geq r_{ess}(P)$ (see e.g. [KM03]). This follows from (5) using $H := \{f \in \mathcal{B}_V : \pi(f) = 0\}$ in (4a)-(4c). The property that P has a spectral gap on $\mathcal{B}_V$ in the recent paper [KM12] corresponds here to the quasi-compactness of P (which is a classical terminology in spectral theory). The spectral gap in [KM12] corresponds to the value $1 - \rho_V(P)$. Then, [KM12, Prop. 1.1]) is another formulation, under ψ-irreducibility and aperiodicity assumptions, of the equivalence of properties (a) and (b) in Proposition 2.1 (see also [KM12, Lem. 2.1]). Details on the proof of Proposition 2.1 are provided in [GHL11]. For general quasi-compact Markov kernels on $\mathcal{B}_V$, the result [Wu04, Th. 4.6] also provides interesting additional material on peripheral eigen-elements. The next subsection completes the previous spectral description by providing bounds for the generalized eigenfunctions associated with eigenvalues λ such that $\delta \leq |\lambda| \leq 1$, with δ given in (**WD**).

2.2 Bound on generalized eigenfunctions of P

Proposition 2.2 *Assume that the weak drift condition* (**WD**) *holds true. If* $\lambda \in \mathbb{C}$ *is such that* $\delta \leq |\lambda| \leq 1$*, with* δ *given in* (**WD**)*, and if* $f \in \mathcal{B}_V \cap \mathrm{Ker}(P - \lambda I)^p$ *for some* $p \in \mathbb{N}^*$*, then there exists* $c \in (0, +\infty)$ *such that*

$$|f| \leq c\, V^{\frac{\ln|\lambda|}{\ln \delta}}\, (1 + \ln V)^{\frac{p(p-1)}{2}}.$$

Thus, if λ is an eigenvalue such that $|\lambda| = 1$, then any associated eigenfunction f is bounded on $\mathbb{X}$. By contrast, if $|\lambda|$ is close to $\delta_V(P)$, then $|f| \leq c\, V^{\beta(\lambda)}$ with $\beta(\lambda)$ close to 1. The proof of Proposition 2.2 is based on the following lemma.

Lemma 2.3 *Let* $\lambda \in \mathbb{C}$ *be such that* $\delta \leq |\lambda| \leq 1$*. Then*

$$\forall f \in \mathcal{B}_V,\ \exists c \in (0, +\infty),\ \forall x \in \mathbb{X}, \quad |\lambda|^{-n(x)} \big|(P^{n(x)} f)(x)\big| \leq c\, V(x)^{\frac{\ln|\lambda|}{\ln \delta}} \tag{6}$$

with, for any $x \in \mathbb{X}$, $n(x) := \big\lfloor \frac{-\ln V(x)}{\ln \delta} \big\rfloor$ *where* $\lfloor \cdot \rfloor$ *denotes the integer part function.*

Proof. First note that the iteration of (**WD**) gives

$$\forall k \geq 1, \quad P^{kN} V \leq \delta^{kN}\, V + d\big(\sum_{j=0}^{k-1} \delta^{jN}\big)\, 1_{\mathbb{X}} \leq \delta^{kN}\, V + \frac{d}{1 - \delta^N}\, 1_{\mathbb{X}}.$$

Let $g \in \mathcal{B}_V$ and $x \in \mathbb{X}$. Using the last inequality, the positivity of P and $|g| \leq \|g\|_V\, V$, we obtain with $b := d/(1 - \delta^N)$:

$$\forall k \geq 1, \quad |(P^{kN} g)(x)| \leq (P^{kN}|g|)(x) \leq \|g\|_V\, (P^{kN} V)(x) \leq \|g\|_V \big(\delta^{kN} V(x) + b\big). \tag{7}$$

The previous inequality is also fulfilled with $k=0$. Next, let $f \in \mathcal{B}_V$ and $n \in \mathbb{N}$. Writing $n = kN + r$, with $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, N-1\}$, and applying (7) to $g := P^r f$, we obtain with $\xi := \max_{0 \le \ell \le N-1} \|P^\ell f\|_V$ (use $P^n f = P^{kN}(P^r f)$):

$$\big|(P^n f)(x)\big| \le \xi\big[\delta^{kN} V(x) + b\big] \le \xi\big[\delta^{-r}\big(\delta^n V(x) + b\big)\big] \le \xi\, \delta^{-N}\big(\delta^n V(x) + b\big). \tag{8}$$

Using the inequality

$$-\frac{\ln V(x)}{\ln \delta} - 1 \le n(x) \le -\frac{\ln V(x)}{\ln \delta}$$

and the fact that $\ln \delta \le \ln |\lambda| \le 0$, Inequality (8) with $n := n(x)$ gives:

$$\begin{aligned}
|\lambda|^{-n(x)}\big|(P^{n(x)} f)(x)\big| \ &\le \ \xi\, \delta^{-N}\bigg(\big(\delta|\lambda|^{-1}\big)^{n(x)}\, V(x) + b\, |\lambda|^{-n(x)}\bigg) \\
&= \xi\, \delta^{-N}\bigg(e^{n(x)(\ln\delta - \ln|\lambda|)}\, e^{\ln V(x)} + b\, e^{-n(x)\ln|\lambda|}\bigg) \\
&\le \ \xi\, \delta^{-N}\bigg(e^{(\frac{\ln V(x)}{\ln \delta}+1)\,(\ln|\lambda| - \ln\delta)}\, e^{\ln V(x)} + b\, e^{\frac{\ln V(x)}{\ln\delta}\ln|\lambda|}\bigg) \\
&= \xi\, \delta^{-N}\bigg(e^{\frac{\ln|\lambda|}{\ln\delta}\ln V(x)}\, e^{\ln|\lambda| - \ln\delta} + b\, V(x)^{\frac{\ln|\lambda|}{\ln\delta}}\bigg) \\
&= \xi\, \delta^{-N}\big(e^{\ln|\lambda| - \ln\delta} + b\big)\, V(x)^{\frac{\ln|\lambda|}{\ln\delta}}.
\end{aligned}$$

This gives Inequality (6) with $c := \xi\, \delta^{-N}(e^{\ln|\lambda| - \ln\delta} + b)$. □

Proof of Proposition 2.2. If $f \in \mathcal{B}_V \cap \operatorname{Ker}(P - \lambda I)$, then $|\lambda|^{-n(x)}|(P^{n(x)} f)(x)| = |f(x)|$, so that (6) gives the expected conclusion when $p = 1$. Next, let us proceed by induction. Assume that the conclusion of Proposition 2.2 holds for some $p \ge 1$. Let $f \in \mathcal{B}_V \cap \operatorname{Ker}(P - \lambda I)^{p+1}$. We can write

$$P^n f = (P - \lambda I + \lambda I)^n f = \lambda^n\, f + \sum_{k=1}^{\min(n,p)} \binom{n}{k} \lambda^{n-k}\, (P - \lambda I)^k f. \tag{9}$$

For $k \in \{1, \ldots, p\}$, we have $f_k := (P - \lambda I)^k f \in \operatorname{Ker}(P - \lambda I)^{p+1-k} \subset \operatorname{Ker}(P - \lambda I)^p$, thus we have from the induction hypothesis :

$$\exists c' \in (0, +\infty),\ \forall k \in \{1, \ldots, p\},\ \forall x \in \mathbb{X}, \quad |f_k(x)| \le c'\, V(x)^{\frac{\ln|\lambda|}{\ln\delta}}\, (1 + \ln V(x))^{\frac{p(p-1)}{2}}. \tag{10}$$

Now, we obtain from (9) (with $n := n(x)$), (10) and Lemma 2.3 that for all $x \in \mathbb{X}$:

$$\begin{aligned}
|f(x)| \ &\le \ |\lambda|^{-n(x)}\big|(P^{n(x)} f)(x)\big| + c'\, V(x)^{\frac{\ln|\lambda|}{\ln\delta}}\, (1 + \ln V(x))^{\frac{p(p-1)}{2}}\, |\lambda|^{-\min(n,p)} \sum_{k=1}^{\min(n,p)} \binom{n(x)}{k} \\
&\le \ c\, V(x)^{\frac{\ln|\lambda|}{\ln\delta}} + c_1\, V(x)^{\frac{\ln|\lambda|}{\ln\delta}}\, (1 + \ln V(x))^{\frac{p(p-1)}{2}}\, n(x)^p \\
&\le \ c_2 V(x)^{\frac{\ln|\lambda|}{\ln\delta}}\, (1 + \ln V(x))^{\frac{p(p-1)}{2} + p}
\end{aligned}$$

with some constants $c_1, c_2 \in (0, +\infty)$ independent of x. This gives the expected result. □

3 Spectral properties of discrete Random Walks

In the sequel, the state space $\mathbb{X}$ is discrete. For the sake of simplicity, we assume that $\mathbb{X} := \mathbb{N}$. Let $P = (P(i,j))_{i,j\in\mathbb{N}^2}$ be a Markov kernel on $\mathbb{N}$. The function $V : \mathbb{N} \to [1, +\infty)$ is assumed to satisfy

$$\lim_n V(n) = +\infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} \frac{(PV)(n)}{V(n)} < \infty.$$

The first focus is on the estimation of $r_{ess}(P)$ from Condition (**WD**).

Proposition 3.1 *Let $\mathbb{X} := \mathbb{N}$. The two following conditions are equivalent:*

(a) Condition (**WD**) *holds with V;*

(b) $L := \inf_{N\geq 1} (\ell_N)^{\frac{1}{N}} < 1$ *where* $\ell_N := \limsup_{n\to+\infty} \dfrac{(P^N V)(n)}{V(n)}$.

In this case, P is power-bounded and quasi-compact on $\mathcal{B}_V$ with $r_{ess}(P) = \delta_V(P) = L$.

The proof of the equivalence $(a) \Leftrightarrow (b)$, as well as the equality $\delta_V(P) = L$, is straightforward (see [GHL11, Cor. 4]). That P is quasi-compact on $\mathcal{B}_V$ under (**WD**) in the discrete case, with $r_{ess}(P) \leq \delta_V(P)$, can be derived from [Wu04] or [HL14] (see Subsection 2.1 and use the fact that the injection from $\mathcal{B}_0$ to $\mathcal{B}_V$ is compact when $\mathbb{X} := \mathbb{N}$ and $\lim_n V(n) = +\infty$). Equality $r_{ess}(P) = \delta_V(P)$ can be proved by combining the results [Wu04, HL14] (see [GHL11, Cor. 1] for details).

In Sections 3 and 4, sequences of the special form $V_\gamma := \{\gamma^n\}_{n\in\mathbb{N}}$ for some $\gamma \in (1, +\infty)$ will be considered. The associated weighted-supremum space $\mathcal{B}_\gamma \equiv \mathcal{B}_{V_\gamma}$ is defined by:

$$\mathcal{B}_\gamma := \big\{\{f(n)\}_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{n\in\mathbb{N}} \gamma^{-n}|f(n)| < \infty\big\}.$$

3.1 Quasi-compactness of RWs with bounded state-dependent increments

Let us fix $c, g, d \in \mathbb{N}^*$, and assume that the kernel P satisfies the following conditions:

$$\forall i \in \{0, \ldots, g-1\}, \quad \sum_{j=0}^{c} P(i,j) = 1; \tag{11a}$$

$$\forall i \geq g, \forall j \in \mathbb{N}, \quad P(i,j) = \begin{cases} a_{j-i}(i) & \text{if } i-g \leq j \leq i+d \\ 0 & \text{otherwise} \end{cases} \tag{11b}$$

where $(a_{-g}(i), \ldots, a_d(i)) \in [0,1]^{g+d+1}$ satisfies $\sum_{k=-g}^{d} a_k(i) = 1$ for all $i \geq g$. This kind of kernels arises, for instance, from time-discretization of Markovian queuing models. Note that more general models and their use in queuing theory are discussed in [KD06]. In particular, conditions for (non) positive recurrence are provided.

Proposition 3.2 *Assume that, for every $k \in \mathbb{Z}$ such that $-g \le k \le d$, $\lim_n a_k(n) = a_k \in [0,1]$, and that*

$$\exists \gamma \in (1,+\infty): \qquad \phi(\gamma) := \sum_{k=-g}^{d} a_k\, \gamma^k < 1. \tag{12}$$

Then P satisfies Condition (**WD**) *with $\delta = \phi(\gamma)$. Moreover P is power-bounded and quasi-compact on $\mathcal{B}_\gamma$ with $r_{ess}(P) = L = \phi(\gamma)$.*

Lemma 3.3 *When a_{-g} and a_d are positive, Condition* (12) *is equivalent to*

$$\sum_{k=-g}^{d} k\, a_k < 0. \tag{\textbf{NERI}}$$

Then, there exists a unique real number $\gamma_0 > 1$ such that $\phi(\gamma_0) = 1$ and

$$\forall \gamma \in (1, \gamma_0), \quad \phi(\gamma) < 1$$

and there is a unique $\widehat{\gamma}$ such that

$$\widehat{\delta} := \phi(\widehat{\gamma}) = \min_{\gamma \in (1,\infty)} \phi(\gamma) = \min_{\gamma \in (1,\gamma_0)} \phi(\gamma) < 1.$$

Condition (**NERI**) means that the expectation of the probability distribution of the random increment is negative. Although the results of the paper on RWs with i.d. bounded increments involving Condition (**NERI**) and $a_{-g}, a_d > 0$ will be valid for $\gamma \in (1, \gamma_0)$, only this value $\widehat{\gamma}$ is considered in the statements. Note that the essential spectral radius $r_{ess}(P_{|\mathcal{B}_{\widehat{\gamma}}})$ of P with respect to $\mathcal{B}_{\widehat{\gamma}}$, which will be denoted by $\widehat{r}_{ess}(P)$ in the sequel, is the smallest value of $r_{ess}(P_{|\mathcal{B}_\gamma})$ on $\mathcal{B}_\gamma$ for $\gamma \in (1, \gamma_0)$. When $\gamma \nearrow \gamma_0$, the essential spectral radius $r_{ess}(P_{|\mathcal{B}_\gamma}) \nearrow 1$ since the space $\mathcal{B}_\gamma$ becomes large. When $\gamma \searrow 1$, then $r_{ess}(P_{|\mathcal{B}_\gamma}) \nearrow 1$ since $\mathcal{B}_\gamma$ becomes close to the space $\mathcal{B}_0$ of bounded functions. In this case, the geometric ergodicity is lost since the RWs are typically not uniformly ergodic (i.e. $V \equiv 1$) due the non quasi-compactness of P on $\mathcal{B}_0$.

Example 1 (State-dependent birth-and-death Markov chains) *When $c = g = d := 1$ in* (11a)-(11b)*, we obtain the standard class of state-dependent birth-and-death Markov chains:*

$$P(0,0) := r_0, \quad P(0,1) := q_0$$
$$\forall n \ge 1,\ P(n,n-1) := p_n, \quad P(n,n) := r_n, \quad P(n,n+1) := q_n,$$

where $(p_0, q_0) \in [0,1]^2, p_0 + q_0 = 1$ and $(p_n, r_n, q_n) \in [0,1]^3, p_n + r_n + q_n = 1$. Assume that:

$$\lim_n p_n := p \quad \lim_n r_n := r, \quad \lim_n q_n := q.$$

If $\gamma \in (1,+\infty)$ is such that $\phi(\gamma) := p/\gamma + r + q\gamma < 1$ then it follows from Proposition 3.2 that $r_{ess}(P) = p/\gamma + r + q\gamma$. The conditions $\gamma > 1$ and $p/\gamma + r + q\gamma < 1$ are equivalent to the following ones (use $r = 1 - p - q$ for (i)):

(i) *either* $q > 0$, $q - p < 0$ *(i.e.* (**NERI**)*) and* $1 < \gamma < \gamma_0 = p/q$;
(ii) *or* $q = 0$, $p > 0$ *and* $\gamma > 1$.

(i) When $p > q > 0$ *and* $1 < \gamma < \gamma_0$*:* P *is power-bounded and quasi-compact on* $\mathcal{B}_\gamma$ *with* $r_{ess}(P) = \phi(\gamma)$. *Set* $\widehat{\gamma} := \sqrt{\gamma_0} = \sqrt{p/q} \in (1, \gamma_0)$. *Then* $\min_{\gamma>1} \phi(\gamma) = \phi(\widehat{\gamma}) = r + 2\sqrt{pq}$ *and the essential spectral radius* $\hat{r}_{ess}(P)$ *on* $\mathcal{B}_{\widehat{\gamma}}$ *satisfies* $\widehat{r}_{ess}(P) = r + 2\sqrt{pq}$.

(ii) When $q := 0, p > 0$ *and* $\gamma > 1$*:* $r_{ess}(P) = \phi(\gamma) = p/\gamma + r$.

Remark 3.1 *If* c *is allowed to be* $+\infty$ *in Condition* (11a)*, that is*

$$\forall i \in \{0, \dots, g-1\}, \quad \sum_{j\geq 0} P(i,j)\gamma^j < \infty, \tag{13}$$

then the conclusions of Proposition 3.2 and Example 1 are still valid under the additional Condition (13).

Proof of Proposition 3.2. Set $\phi_n(\gamma) := \sum_{k=-g}^{d} a_k(n)\, \gamma^k$. We have $(PV_\gamma)(n) = \phi_n(\gamma)V_\gamma(n)$ for each $n \geq g$. Thus $\ell_1 = \lim_n \phi_n(\gamma) = \phi(\gamma)$. Now assume that $\ell_{N-1} := \lim_n (P^{N-1}V)(n)/V(n) = \phi(\gamma)^{N-1}$ for some $N \geq 1$. Since

$$\forall i \geq Ng, \quad (P^N V)(i) = \sum_{j=-g}^{d} a_j(i)\, (P^{N-1}V)(i+j)$$

we obtain

$$\frac{(P^N V)(i)}{V(i)} = \sum_{j=-g}^{d} a_j(i)\gamma^j\, \frac{(P^{N-1}V)(i+j)}{\gamma^{i+j}} \xrightarrow[i \to +\infty]{} \phi(\gamma)\, \phi(\gamma)^{N-1}.$$

Hence $\ell_N = \phi(\gamma)^N$, and $\phi(\gamma) = L = r_{ess}(P)$ from Proposition 3.1. $\square$

Proof of Lemma 3.3. Since the second derivative of ϕ is positive on $(0, +\infty)$, ϕ is convex on $(0, +\infty)$. When a_{-g} and a_d are positive then $\lim_{t\to 0^+} \phi(t) = \lim_{t\to+\infty} \phi(t) = +\infty$ and, since $\phi(1) = 1$, Condition (12) is equivalent to $\phi'(1) < 0$, that is (**NERI**). The other properties of $\phi(\cdot)$ are immediate. $\square$

3.2 Spectral analysis of RW with i.d. bounded increments

Let $P := (P(i,j))_{(i,j)\in\mathbb{N}^2}$ be the transition kernel of a RW with i.d. bounded increments. Specifically we assume that there exist some positive integers $c, g, d \in \mathbb{N}^*$ such that

$$\forall i \in \{0, \dots, g-1\}, \quad \sum_{j=0}^{c} P(i,j) = 1; \tag{14a}$$

$$\forall i \geq g, \forall j \in \mathbb{N}, \quad P(i,j) = \begin{cases} a_{j-i} & \text{if } \ i - g \leq j \leq i + d \\ 0 & \text{otherwise.} \end{cases} \tag{14b}$$

$$(a_{-g}, \dots, a_d) \in [0,1]^{g+d+1} : a_{-g} > 0, \ a_d > 0, \ \sum_{k=-g}^{d} a_k = 1. \tag{14c}$$

Let us assume that Condition (**NERI**) holds. We know from Lemma 3.3 and Proposition 3.2 that P is quasi-compact on $\mathcal{B}_{\widehat{\gamma}}$ with

$$\widehat{r}_{ess}(P) = \widehat{\delta} := \phi(\widehat{\gamma}) < 1$$

where $\phi(\cdot)$ is given by (12).

For any $\lambda \in \mathbb{C}$, we denote by $E_\lambda(\cdot)$ the following polynomial of degree $N := d + g$

$$\forall z \in \mathbb{C}, \quad E_\lambda(z) := z^g\big(\phi(z) - \lambda\big) = \sum_{k=-g}^{d} a_k z^{g+k} - \lambda\, z^g,$$

and by $\mathcal{E}_\lambda$ the set of complex roots of $E_\lambda(\cdot)$. Since $E_\lambda(0) = a_{-g} > 0$, we have for any $\lambda \in \mathbb{C}$:

$$z \in \mathcal{E}_\lambda \iff \mathbb{E}_\lambda(z) = 0 \iff \lambda = \phi(z).$$

The next proposition investigates the eigenvalues of P on $\mathcal{B}_{\widehat{\gamma}}$ which belong to the annulus

$$\Lambda := \{\lambda \in \mathbb{C} : \widehat{\delta} < |\lambda| < 1\}.$$

To that effect, for any $\lambda \in \Lambda$, we introduce the following subset $\mathcal{E}_\lambda^-$ of $\mathcal{E}_\lambda$

$$\mathcal{E}_\lambda^- := \big\{z \in \mathbb{C} :\ E_\lambda(z) = 0,\ |z| < \widehat{\gamma}\big\}.$$

If $\mathcal{E}_\lambda^- = \emptyset$, we set $N(\lambda) := 0$. If $\mathcal{E}_\lambda^- \neq \emptyset$, then $N(\lambda)$ is defined as

$$N(\lambda) := \sum_{z \in \mathcal{E}_\lambda^-} m_z,$$

where m_z denotes the multiplicity of z as root of $E_\lambda(\cdot)$. Finally, for any $z \in \mathbb{C}$, we set $z^{(1)} := \{z^n\}_{n \in \mathbb{N}}$, and for any $k \geq 2$, $z^{(k)} \in \mathbb{C}^{\mathbb{N}}$ is defined by:

$$\forall n \in \mathbb{N}, \quad z^{(k)}(n) := n(n-1)\cdots(n-k+2)\, z^{n-k+1}.$$

Proposition 3.4 *Assume that Assumptions* (14a)-(14c) *and* (**NERI**) *hold true. Then*

$$\exists\, \eta \geq 1,\ \forall \lambda \in \Lambda, \quad N(\lambda) = \eta.$$

Moreover the two following assertions are equivalent:

(i) $\lambda \in \Lambda$ *is an eigenvalue of* P *on* $\mathcal{B}_{\widehat{\gamma}}$.

(ii) There exists a nonzero $\{\alpha_{\lambda,z,k}\}_{z \in \mathcal{E}_\lambda^-, 1 \leq k \leq m_z} \in \mathbb{C}^\eta$ *such that*

$$f := \sum_{z \in \mathcal{E}_\lambda^-} \sum_{k=1}^{m_z} \alpha_{\lambda,z,k}\, z^{(k)} \in \mathbb{C}^{\mathbb{N}} \tag{15}$$

satisfies the boundary equations: $\forall i = 0, \ldots, g-1,\ \lambda f(i) = (Pf)(i)$.

The first step of the elimination procedure of Section 4 is to plug f of the form (15) in the boundary equations. This gives a linear system in $\alpha_{\lambda,z,k}$. Since Λ is infinite, that $N(\lambda)$ does not depend on λ is crucial to initialize this procedure. To specify the value of η, it is sufficient to compute $N(\lambda)$ for some (any) $\lambda \in \Lambda$.

Remark 3.2 *Under Condition* (**NERI**), $\phi(\cdot)$ *is strictly decreasing from* $(1, \widehat{\gamma})$ *to* $(\widehat{\delta}, 1)$, *so that we have:* $\forall \lambda \in (\widehat{\delta}, 1),\ \phi^{-1}(\lambda) \in (1, \widehat{\gamma})$. *Since* $\phi^{-1}(\lambda) \in \mathcal{E}_\lambda$, *we obtain*

$$\forall \lambda \in (\widehat{\delta}, 1), \quad N(\lambda) \geq 1. \tag{16}$$

Remark 3.3 *Let Condition* (**NERI**) *be satisfied. Set* $\mathcal{E}_\lambda^+ := \{z \in \mathbb{C} :\ E_\lambda(z) = 0,\ |z| > \widehat{\gamma}\}$. *Then*

$$\forall \lambda \in \Lambda, \quad \mathcal{E}_\lambda = \mathcal{E}_\lambda^- \sqcup \mathcal{E}_\lambda^+.$$

In other words, for any $\lambda \in \Lambda$, $E_\lambda(\cdot)$ *has no root of modulus* $\widehat{\gamma}$. *Indeed, consider* $\lambda \in \Lambda$, $z \in \mathcal{E}_\lambda$, *and assume that* $|z| = \widehat{\gamma}$. *Since* $\lambda = \phi(z)$, *we obtain the inequality* $|\lambda| \leq \phi(|z|) = \phi(\widehat{\gamma})$ *which is impossible since* $\phi(\widehat{\gamma}) = \widehat{\delta}$ *and* $\lambda \in \Lambda$.

Remark 3.4 *Assertion* (*ii*) *of Proposition 3.4 does not mean that the dimension of the eigenspace* $\mathrm{Ker}(P - \lambda I)$ *associated with* λ *is* η. *We shall see in Subsection 4.2 that we can have* $\eta = 2$ *when* $g = 2$, $d = 1$ *and* $c = 2$ *in* (14a)-(14c), *while* $\dim \mathrm{Ker}(P - \lambda I) \leq 1$ *since* $Pf = \lambda f$ *and* $f(0) = 0$ *clearly imply* $f = 0$ *(by induction).*

The following surprising lemma, based on Remark 3.3, is used to derive Proposition 3.4.

Lemma 3.5 *Under Condition* (**NERI**), *the function* $N(\cdot)$ *is constant on* Λ.

Proof. Since Λ is connected and $N(\cdot)$ is $\mathbb{N}$-valued, it suffices to prove that $N(\cdot)$ is continuous on Λ. Note that the set $\cup_{\lambda \in \Lambda} \mathcal{E}_\lambda$ is bounded in $\mathbb{C}$ since the coefficients of $E_\lambda(\cdot)$ are obviously uniformly bounded in $\lambda \in \Lambda$. Now let $\lambda \in \Lambda$ and assume that $N(\cdot)$ is not continuous at λ. Then there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \in \Lambda^{\mathbb{N}}$ such that $\lim_n \lambda_n = \lambda$ and

(a) either: $\forall n \geq 0,\ N(\lambda_n) \geq N(\lambda) + 1$,

(b) or: $\forall n \geq 0,\ N(\lambda_n) \leq N(\lambda) - 1$.

For any $n \geq 0$, let us denote the roots of $E_{\lambda_n}(\cdot)$ by $z_1(\lambda_n), \ldots, z_N(\lambda_n)$, and suppose for convenience that they are listed by increasing modulus, and by increasing argument when they have the same modulus. Applying Remark 3.3 to λ_n, we obtain:

$$\forall i \in \{1, \ldots, N(\lambda_n)\},\ |z_i(\lambda_n)| < \widehat{\gamma} \quad \text{and} \quad \forall i \in \{N(\lambda_n) + 1, \ldots, N\},\ |z_i(\lambda_n)| > \widehat{\gamma}.$$

Up to consider a subsequence, we may suppose that, for every $1 \leq i \leq N$, the sequence $\{z_i(\lambda_n)\}_{n \in \mathbb{N}}$ converges to some $z_i \in \mathbb{C}$. Note that

$$\mathcal{E}_\lambda = \{z_1, z_2, \ldots, z_N\}$$

where z_i is repeated in this list with respect to its multiplicity m_{z_i}, since

$$\forall z \in \mathbb{C}, \quad E_\lambda(z) = \lim_n E_{\lambda_n}(z) = \lim_n a_d \prod_{i=1}^{N}(z - z_i(\lambda_n)) = a_d \prod_{i=1}^{N}(z - z_i).$$

In case (a), we have

$$\forall n \geq 0, \qquad |z_1(\lambda_n)| < \widehat{\gamma}, \ \ldots, \ |z_{N(\lambda)+1}(\lambda_n)| < \widehat{\gamma}.$$

When $n \to +\infty$, this gives using Remark 3.3:

$$|z_1| < \widehat{\gamma}\ , \ldots, |z_{N(\lambda)+1}| < \widehat{\gamma}.$$

Thus at least $N(\lambda)+1$ roots of $E_\lambda(\cdot)$ (counted with their multiplicity) are of modulus strictly less than $\widehat{\gamma}$: this contradicts the definition of $N(\lambda)$.
In case (b), we have

$$\forall n \geq 0, \qquad |z_{N(\lambda)}(\lambda_n)| > \widehat{\gamma}, \ |z_{N(\lambda)+1}(\lambda_n)| > \widehat{\gamma}, \ \ldots, \ |z_N(\lambda_n)| > \widehat{\gamma},$$

and this gives similarly when $n \to +\infty$

$$|z_{N(\lambda)}| > \widehat{\gamma}, \ |z_{N(\lambda)+1}| > \widehat{\gamma}, \ldots, |z_N| > \widehat{\gamma}.$$

Thus at least $N - N(\lambda) + 1$ roots of $E_\lambda(\cdot)$ (counted with their multiplicity) are of modulus strictly larger than $\widehat{\gamma}$. This contradicts the definition of $N(\lambda)$. $\square$

Proof of Proposition 3.4. From Lemma 3.5 and (16), we obtain: $\forall \lambda \in \Lambda, \ N(\lambda) = \eta$ for some $\eta \geq 1$. Now we prove the implication $(i) \Rightarrow (ii)$. Let $\lambda \in \Lambda$ be any eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ and let $f := \{f(n)\}_{n \in \mathbb{N}}$ be a nonzero sequence in $\mathcal{B}_{\widehat{\gamma}}$ satisfying $Pf = \lambda f$. In particular f satisfies the following equalities

$$\forall i \geq g, \quad \lambda\, f(i) = \sum_{j=i-g}^{i+g} a_{j-i} f(j). \tag{17}$$

Since the characteristic polynomial associated with these recursive formulas is $E_\lambda(\cdot)$, there exists $\{\alpha_{\lambda,z,k}\}_{z \in \mathcal{E}_\lambda, 1 \leq k \leq m_z} \in \mathbb{C}^\eta$ such that

$$f = \sum_{z \in \mathcal{E}_\lambda} \sum_{k=1}^{m_z} \alpha_{\lambda,z,k}\, z^{(k)} \in \mathbb{C}^{\mathbb{N}}$$

where m_z denotes the multiplicity of $z \in \mathcal{E}_\lambda$. Next, since $|f| \leq C\, V_{\widehat{\gamma}}$ for some $C > 0$ (i.e. $f \in \mathcal{B}_{\widehat{\gamma}}$), it can be easily seen that $\alpha_{\lambda,z,k} = 0$ for every $z \in \mathcal{E}_\lambda$ such that $|z| > \widehat{\gamma}$ and for every $k = 1, \ldots, m_z$: : first delete α_{λ,z,m_z} for z of maximum modulus and for m_z maximal if there are several z of maximal modulus (to that effect, divide f by $n(n-1)\cdots(n-m_z+2)\, z^{n-m_z+1}$ and use $|f| \leq CV_{\widehat{\gamma}}$). Therefore f is of the form (15), and it satisfies the boundary equations in (ii) since $Pf = \lambda f$ by hypothesis.

To prove the implication $(ii) \Rightarrow (i)$, note that any $f := \{f(n)\}_{n \in \mathbb{N}}$ of the form (15) belongs to $\mathcal{B}_{\widehat{\gamma}}$ and satisfies (17) since $\mathcal{E}_\lambda^- \subset \mathcal{E}_\lambda$. If moreover f is non zero and satisfies the boundary equations, then $Pf = \lambda f$. This gives (i). $\square$

We conclude this study with an additional refinement of Proposition 3.4. For any $\lambda \in \Lambda$, let us define the set $\mathcal{E}^-_{\lambda,\tau}$ as follows:

$$\mathcal{E}^-_{\lambda,\tau} := \{z \in \mathbb{C} : \; E_\lambda(z) = 0, \; |z| < \widehat{\gamma}^\tau\} \quad \text{with} \quad \tau \equiv \tau(\lambda) := \frac{\ln|\lambda|}{\ln \widehat{\delta}}.$$

Moreover define the associated function $N'(\cdot)$ by

$$N'(\lambda) := \sum_{z \in \mathcal{E}^-_{\lambda,\tau}} m_z,$$

where m_z is the multiplicity of z as root of $E_\lambda(\cdot)$ (with the convention $N'(\lambda) = 0$ if $\mathcal{E}^-_{\lambda,\tau} = \emptyset$).

Lemma 3.6 *Assume that* $P := (P(i,j))_{(i,j)\in\mathbb{N}^2}$ *satisfies Conditions* (14a)-(14c) *and* (**NERI**). *Moreover assume that*

$$\forall t \in (1, \widehat{\gamma}), \quad \phi(t) < t^{\ln \widehat{\delta}/\ln \widehat{\gamma}} \tag{18}$$

Then the function $N'(\cdot)$ *is constant on* Λ*:* $\exists\, \eta' \geq 1, \; \forall \lambda \in \Lambda, \; N'(\lambda) = \eta'$.

From Lemma 3.6, all the assertions of Proposition 3.4 are still valid when η and $\mathcal{E}^-_\lambda$ are replaced with η' and $\mathcal{E}^-_{\lambda,\tau}$ respectively. That $\mathcal{E}^-_\lambda$ may be replaced with $\mathcal{E}^-_{\lambda,\tau}$ in (15) follows from Proposition 2.2. Consequently, under the additional condition $\eta' \leq g$, the elimination procedure of Section 4 may be adapted by using Lemma 3.6. Since $\eta' \leq \eta$, the resulting procedure is computationally interesting when g or d are large.

Remark 3.5 *Condition* (18) *is the additional assumption in Lemma 3.6 with respect to Lemma 3.5. Since* ϕ *is strictly decreasing on* $(1, \widehat{\gamma})$ *under Condition* (**NERI**)*, Condition* (18) *is equivalent to the following one*

$$\forall z \in (1, \widehat{\gamma}), \quad z < \widehat{\gamma}^{\ln \phi(z)/\ln \widehat{\delta}}. \tag{19}$$

Indeed, for every $t \in (1, \widehat{\gamma})$*, we have* $u := t^{\ln \widehat{\delta}/\ln \widehat{\gamma}} \in (\widehat{\delta}, 1)$ *and* $z := \phi^{-1}(u) \in (1, \widehat{\gamma})$*. Hence*

$$(18) \iff \forall u \in (\widehat{\delta}, 1), \; \phi\big(\widehat{\gamma}^{\ln u/\ln \widehat{\delta}}\big) < u \iff (19). \tag{20}$$

Therefore, under Condition (18)*, for any* $\lambda \in (\widehat{\delta}, 1)$ *we have* $\mathcal{E}^-_{\lambda,\tau} \neq \emptyset$ *since* $z = \phi^{-1}(\lambda)$ *satisfies* $z < \widehat{\gamma}^{\tau(\lambda)}$ *from* (19)*.*

Proof of Lemma 3.6. The proof is similar to that of Lemma 3.5. Under Condition (18), Remark 3.3 extends as follows:

$$\mathcal{E}_\lambda = \mathcal{E}^-_{\lambda,\tau} \sqcup \big(\mathcal{E}_\lambda \cap \{z \in \mathbb{C} : \; |z| > \widehat{\gamma}^\tau\}\big). \tag{21}$$

Indeed, consider $\lambda \in \Lambda$ and $z \in \mathcal{E}_\lambda$ such that $|z| = \widehat{\gamma}^\tau$. Since $\lambda = \phi(z)$, we have $|\lambda| \leq \phi(|z|)$, thus $|\lambda| \leq \phi(\widehat{\gamma}^\tau)$. This inequality contradicts Condition (18) (use the definition of τ and the second equivalence in (20) with $u := |\lambda|$). Next, using (21) and the continuity of $\tau(\cdot)$, Lemma 3.5 easily extends to the function $N'(\cdot)$. $\square$

4 Convergence rate for RWs with i.d. bounded increments

Let us recall that any RW with i.d. bounded increments defined by (14a)-(14c) and satisfying (**NERI**), has an invariant probability measure π on $\mathbb{N}$ such $\pi(V_{\widehat{\gamma}}) < \infty$ where $V_{\widehat{\gamma}} := \{\widehat{\gamma}^n\}_{n\in\mathbb{N}}$ and $\widehat{\gamma}$ is defined in Lemma 3.3. Indeed $\widehat{\delta} := \phi(\widehat{\gamma}) < 1$ so that Condition (**WD**) holds with $V_{\widehat{\gamma}}$ from Proposition 3.2. The expected conclusions on π can be deduced from the first statement of [GHL11, Cor 5]. Note that, from Lemma 3.3, the previous fact is valid for any $\gamma \in (1, \gamma_0)$ in place of $\widehat{\gamma}$.

The $V_{\widehat{\gamma}}$-geometric ergodicity of the RW may be studied using Proposition 2.1. Next we can derive from Proposition 3.4 an effective procedure to compute the rate of convergence with respect to $\mathcal{B}_{\widehat{\gamma}}$ (see (2)), that is denoted by $\widehat{\rho}(P)$. The most favorable case for initializing the procedure (see (24) and (26)) is to assume that for some (any) $\lambda \in \Lambda$

$$\eta := N(\lambda) \leq g. \tag{22}$$

- *First step: checking Condition* (22). From Lemma 3.5, computing η and testing $\eta \leq g$ of Assumption (22) can be done by analyzing the roots of $E_\lambda(\cdot)$ for some (any) $\lambda \in \Lambda$.
- *Second step: linear and polynomial eliminations.* This second step consists in applying some linear and (successive) polynomial eliminations in order to find a finite set $\mathcal{Z} \subset \Lambda$ containing all the eigenvalues of P on $\mathcal{B}_{\widehat{\gamma}}$ in Λ. Conversely, the elements of $\mathcal{Z}$ providing eigenvalues of P on $\mathcal{B}_{\widehat{\gamma}}$ can be identified using Condition (*ii*) of Proposition 3.4. Note that the explicit computation of the roots of $E_\lambda(\cdot)$ is only required for the elements λ of the finite set $\mathcal{Z}$. This is detailed in Corollary 4.1.

Under the assumptions of Proposition 3.4, we define the set

$$\mathcal{M} := \big\{(m_1, \ldots, m_s) \in \{1, \ldots, s\}^s : s \in \{1, \ldots, \eta\}, m_1 \leq \ldots \leq m_s \text{ and } \sum_{i=1}^{s} m_i = \eta\big\}.$$

Note that $\mathcal{M}$ is a finite set and that, for every $\lambda \in \Lambda$, there exists a unique $\mu \in \mathcal{M}$ such that the set $\mathcal{E}_\lambda^-$ is composed of s distinct roots of $E_\lambda(\cdot)$ with multiplicity $m_1, \ldots, m_s$ respectively.

Corollary 4.1 *Assume that Assumptions* (14a)-(14c) *and* (**NERI**) *hold true. Set* $\ell := \binom{g}{\eta}$. *Then there exist a family of polynomials functions* $\{\mathcal{R}_{\mu,k}, \mu \in \mathcal{M}, 1 \leq k \leq \ell\}$, *with coefficients only depending on* μ *and on the transition probabilities* $P(i,j)$, *such that the following assertions hold true for any* $\mu \in \mathcal{M}$.

(i) Let $\lambda \in \Lambda$ *be an eigenvalue of* P *on* $\mathcal{B}_{\widehat{\gamma}}$ *such that, for some* $s \in \{1, \ldots, \eta\}$, *the set* $\mathcal{E}_\lambda^-$ *is composed of* s *roots of* $E_\lambda(\cdot)$ *with multiplicity* $m_1, \ldots, m_s$ *respectively. Then*

$$\mathcal{R}_{\mu,1}(\lambda) = 0, \ldots, \mathcal{R}_{\mu,\ell}(\lambda) = 0. \tag{23}$$

(ii) Conversely, let $\lambda \in \Lambda$ *satisfying (23) such that, for some* $s \in \{1, \ldots, \eta\}$, *the set* $\mathcal{E}_\lambda^-$ *is composed of* s *roots of* $E_\lambda(\cdot)$ *with multiplicity* $m_1, \ldots, m_s$ *respectively. Then a necessary and sufficient condition for* λ *to be an eigenvalue of* P *on* $\mathcal{B}_{\widehat{\gamma}}$ *is that* λ *satisfies Condition* (*ii*) *of Proposition 3.4.*

Proof. Assertion *(ii)* follows from Proposition 3.4. To prove *(i)*, first assume for convenience that $\eta = g$ and that $\lambda \in \Lambda$ is an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ such that the associated set $\mathcal{E}_\lambda^-$ contains η distinct roots $z_1, \dots, z_\eta$ of $E_\lambda(\cdot)$ with multiplicity one. We know from Proposition 3.4 that there exists $f := \{f(n)\}_{n\in\mathbb{N}} \neq 0$ of the form

$$f = \sum_{i=1}^{\eta} \alpha_i \, z_i^{(1)}$$

which satisfies the $g = \eta$ boundary equations: $\forall i = 0, \dots, \eta - 1,\ \lambda f(i) = (Pf)(i)$. In other words the linear system provided by these η equations has a nonzero solution $(\alpha_i)_{1\le i\le \eta} \in \mathbb{C}^\eta$. Therefore the associated determinant is zero: this leads to a polynomial equation of the form

$$P_{0,1}(\lambda, z_1, \dots, z_\eta) = 0. \tag{24}$$

Since this polynomial is divisible by $\prod_{i\neq j}(z_i - z_j)$, Equation (24) is equivalent to

$$P_0(\lambda, z_1, \dots, z_\eta) = 0 \qquad \text{with } P_0(\lambda, z_1, \dots, z_\eta) = \frac{P_{0,1}(\lambda, z_1, \dots, z_\eta)}{\prod_{i\neq j}(z_i - z_j)}. \tag{25}$$

Note that the coefficients of P_0 only depend on the $P(i,j)$'s.

Next, z_η is a common root of the polynomials $P_0(\lambda, z_1, \dots, z_{\eta-1}, z)$ and $E_\lambda(z)$ with respect to the variable z : this leads to the following necessary condition

$$P_1(\lambda, z_1, \dots, z_{\eta-1}) := \mathrm{Res}_{z_\eta}(P_0, E_\lambda) = 0$$

where $\mathrm{Res}_{z_\eta}(P_0, E_\lambda)$ denotes the resultant of the two polynomials P_0 and E_λ corresponding to the elimination of the variable z_η. Again the coefficients of P_1 only depend on the $P(i,j)$'s. Next, considering the common root $z_{\eta-1}$ of the polynomials $P_1(\lambda, z_1, \dots, z_{\eta-2}, z)$ and $E_\lambda(z)$ leads to the elimination of the variable $z_{\eta-1}$

$$P_2(\lambda, z_1, \dots, z_{\eta-2}) := \mathrm{Res}_{z_{\eta-1}}(P_1, E_\lambda) = 0.$$

Repeating this method, we obtain that a necessary condition for λ to be an eigenvalue of P is $\mathcal{R}(\lambda) = 0$ where $\mathcal{R}$ is some polynomial with coefficients only depending on the $P(i,j)$'s.

Now let us consider the case when $\eta < g$, $s \in \{1, \dots, \eta\}$, and $\lambda \in \Lambda$ is assumed to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ such that the associated set $\mathcal{E}_\lambda^-$ contains s distinct roots of $E_\lambda(\cdot)$ with respective multiplicity $m_1, \dots, m_s$ satisfying $\sum_{i=1}^s m_i = \eta$. Then the elimination (by using determinants) of $(\alpha_{\lambda,z,\ell}) \in \mathbb{C}^\eta$ provided by the linear system of Proposition 3.4, leads to $\ell := \binom{g}{\eta}$ polynomial equations

$$P_{0,\mu,1}(\lambda, z_1, \dots, z_\eta) = 0,\ \dots,\ P_{0,\mu,\ell}(\lambda, z_1, \dots, z_\eta) = 0. \tag{26}$$

As in the case $\eta = g$, these polynomials are replaced in the sequel by the polynomials obtained by division of the $P_{0,\mu,k}$'s by $\prod_{i\neq j}(z_i - z_j)^{n_{i,j}}$ where $n_{i,j} := \min(m_i, m_j)$.

The successive polynomial eliminations of $z_\eta, \dots, z_1$ can be derived as above from each polynomial equation $P_{0,\mu,k}(\lambda, z_1, \dots, z_\eta) = 0$. This gives ℓ polynomial equations

$$\mathcal{R}_{\mu,1}(\lambda) = 0\ , \dots,\ \mathcal{R}_{\mu,\ell}(\lambda) = 0.$$

Satisfying this set of polynomial equations is a necessary condition for λ to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$. Finally the polynomial functions $\mathcal{R}_{\mu,1},\ldots\mathcal{R}_{\mu,\ell}$ depend on the $P(i,j)$'s and also on $(m_1,\ldots,m_s)$, since the linear system used to eliminate $(\alpha_{\lambda,k,\ell})\in\mathbb{C}^\eta$ involves coefficients $i(i-1)\cdots(i-k+1)$ for some finitely many integers i and for $k=1,\ldots,m_i$ $(i=1,\ldots,s)$. $\square$

To compute $\widehat{\rho}(P)$, we define the following (finite and possibly empty) sets:

$$\forall\mu\in\mathcal{M},\quad \Lambda_\mu:=\big\{\lambda\in\Lambda:\ \mathcal{R}_{\mu,1}(\lambda)=0\ ,\ldots,\ \mathcal{R}_{\mu,\ell}(\lambda)=0\big\}.$$

Let us denote by $\mathcal{Z}$ the (finite and possibly empty) set composed of all the complex numbers $\lambda\in\cup_{\mu\in\mathcal{M}}\Lambda_\mu$ such that Condition (ii) of Proposition 3.4 holds true.

Corollary 4.2 *Assume that Assumptions* (14a)-(14c) *and* (**NERI**) *hold true and that P is irreducible and aperiodic. Then*

$$\widehat{\rho}(P)=\max\big(\widehat{\delta},\max\{|\lambda|,\ \lambda\in\mathcal{Z}\}\big)\quad \textit{where } \widehat{\delta}:=\phi(\widehat{\gamma}).$$

Proof. Under the assumptions on P, we know from Proposition 2.1 that the RW is $V_{\widehat{\gamma}}$-geometrically ergodic. Since $\widehat{r}_{ess}(P)=\widehat{\delta}$ from Proposition 3.2, the corollary follows from Corollary 4.1 and from Proposition 2.1 applied either with any r_0 such that $\widehat{\delta}<r_0<\min\{|\lambda|,\ \lambda\in\mathcal{Z}\}$ if $\mathcal{Z}\neq\emptyset$, or with any r_0 such that $\widehat{\delta}<r_0<1$ if $\mathcal{Z}=\emptyset$. $\square$

Remark 4.1 *When $\eta\geq 2$ and $\mu:=(m_1,\ldots,m_s)$ with $s<\eta$, the set Λ_μ used in Corollary 4.2 may be reduced. For the sake of simplicity, this fact has been omitted in Corollary 4.2, but it is relevant in practice. Actually, when $s<\eta$, the part (ii) of Corollary 4.1 can be specified since it requires that $E_\lambda(\cdot)$ admits roots of multiplicity ≥ 2. This involves some additional necessary conditions on λ derived from some polynomial eliminations with respect to the derivatives of $E_\lambda(\cdot)$.*

For instance, in case $g=2$, $\eta=2$, $s=1$ (thus $\mu:=(2)$), a necessary condition on λ for $E_\lambda(\cdot)$ to have a double root is that $E_\lambda(\cdot)$ and $E'_\lambda(\cdot)$ admits a common root. This leads to

$$Q(\lambda):=\mathrm{Res}_z\big(E_\lambda,E'_\lambda\big)=0.$$

Consequently, if $g=2$ and $\eta=2$ (thus $\ell:=1$), then Condition (ii) of Proposition 3.4 can be tested in case $s=1$ by using the following finite set

$$\Lambda'_\mu:=\Lambda_\mu\cap\{\lambda\in\Lambda:\ Q(\lambda)=0\}.$$

In general Λ'_μ is strictly contained in Λ_μ. Even Λ'_μ may be empty while Λ_μ is not (see Subsection 4.2).

Proposition 3.4 and the above elimination procedure obviously extend to any $\gamma\in(1,\gamma_0)$ in place of $\widehat{\gamma}$, where γ_0 is given in Lemma 3.3. Of course $\widehat{\delta}=\phi(\widehat{\gamma})$ is then replaced by $\delta=\phi(\gamma)$.

4.1 RWs with $g = d := 1$: birth-and-death Markov chains

Let $p, q, r \in [0, 1]$ be such that $p + r + q = 1$, and let P be defined by

$$\begin{aligned} & P(0,0) \in (0,1), P(0,1) = 1 - P(0,0) \\ & \forall n \geq 1,\ P(n, n-1) := p, \quad P(n,n) := r, \quad P(n, n+1) := q \quad \text{with } 0 < q < p. \end{aligned} \tag{27}$$

Note that $a_{-1} := p, a_1 := q > 0$ and (**NERI**) holds true. We have $\gamma_0 = p/q \in (1, +\infty)$ and $\widehat{\gamma} := \sqrt{p/q} \in (1, +\infty)$ is such that $\widehat{\delta} := \min_{\gamma > 1} \phi(\gamma) = \phi(\widehat{\gamma}) < 1$ (see Lemma 3.3). Let $V_{\widehat{\gamma}} := \{\widehat{\gamma}^n\}_{n \in \mathbb{N}}$ and its associated weighted-supremum space $\mathcal{B}_{\widehat{\gamma}}$. Here we have

$$\widehat{r}_{ess}(P) = \widehat{\delta} = r + 2\sqrt{pq}.$$

Proposition 4.1 *Let P be defined by Conditions* (27)*. The boundary transition probabilities are denoted by $P(0,0) := a, P(0,1) := 1 - a$ for some $a \in (0,1)$. Then P is $V_{\widehat{\gamma}}$-geometrically ergodic. Furthermore, defining $a_0 := 1 - q - \sqrt{pq}$, the convergence rate $\widehat{\rho}(P)$ of P with respect to $\mathcal{B}_{\widehat{\gamma}}$ is given by:*

- *when $a \in (a_0, 1)$:*
$$\widehat{\rho}(P) = r + 2\sqrt{pq}\,; \tag{28}$$
- *when $a \in (0, a_0]$:*
  - *(a) in case $2p \leq \big(1 - q + \sqrt{pq}\big)^2$:*
$$\widehat{\rho}(P) = r + 2\sqrt{pq}\,; \tag{29}$$
  - *(b) in case $2p > \big(1 - q + \sqrt{pq}\big)^2$, set $a_1 := p - \sqrt{pq} - \sqrt{r\big(r + 2\sqrt{pq}\big)}$:*
$$\widehat{\rho}(P) = \left| a + \frac{p(1-a)}{a - 1 + q} \right| \quad \text{when } a \in (0, a_1] \tag{30a}$$
$$\widehat{\rho}(P) = r + 2\sqrt{pq} \qquad \text{when } a \in [a_1, a_0). \tag{30b}$$

When $r := 0$, such results have been obtained in [RT99, Bax05, LT96] by using various methods involving conditions on a (see the end of Introduction). Let us specify the above formulas in case $r := 0$. We have $a_0 = a_1 = p - \sqrt{pq} = (p - q)/(1 + \sqrt{q/p})$, and it can be easily checked that $2p > (1 - q + \sqrt{pq})^2$. Then the properties (28), (30a), (30b) then rewrite as: $\widehat{\rho}(P) = (pq + (a-p)^2)/|a - p|$ when $a \in (0, a_0]$, and $\widehat{\rho}(P) = 2\sqrt{pq}$ when $a \in (a_0, 1)$.

Proof. We apply the elimination procedure of Section 4. Then $\Lambda := \{\lambda \in \mathbb{C} : \widehat{\delta} < |\lambda| < 1\}$ with $\widehat{\delta} := r + 2\sqrt{pq}$. The characteristic polynomial $E_\lambda(\cdot)$ is

$$E_\lambda(z) := qz^2 + (r - \lambda)z + p.$$

A simple study of the graph of $\phi(t) := p/t + r + qt$ on $\mathbb{R} \setminus \{0\}$ shows that, for any $\lambda \in (\widehat{\delta}, 1)$, the equation $\phi(z) = \lambda$ (ie. $E_\lambda(z) = 0$) admits a solution in $(1, \widehat{\gamma})$ and another one in $(\widehat{\gamma}, +\infty)$, so that $N(\lambda) = 1$. It follows from Proposition 3.4 that $\eta = 1$. Thus the linear elimination used in Corollary 4.1 is here trivial. Indeed, a necessary condition for $f := \{z^n\}_{n\in\mathbb{N}}$ to satisfy $Pf = \lambda f$ is obtained by eliminating the variable z with respect to the boundary equation $(Pf)(0) = \lambda f(0)$, namely $P_0(\lambda, z) := a + (1-a)z = \lambda$, and Equation $E_\lambda(z) = 0$. This leads to

$$P_1(\lambda, z) := Res_z(P_0, E_\lambda) = (1-\lambda)\big[(\lambda-a)(1-a-q) + p(1-a)\big]. \tag{31}$$

In the special case $a = 1-q$, the only solution of (31) is $\lambda = 1$. Corollary 4.2 then gives $\widehat{\rho}(P) = r + 2\sqrt{pq}$.

Now assume that $a \neq 1-q$. Then $\lambda = 1$ is a solution of (31) and the other solution of (31), say $\lambda(a)$, and the associated complex number, say $z(a)$, are given by the following formulas (use $a + (1-a)z = \lambda$ to obtain $z(a)$):

$$\lambda(a) := a + \frac{p(1-a)}{a-1+q} \in \mathbb{R} \quad \text{and} \quad z(a) := \frac{p}{a+q-1} \in \mathbb{R}.$$

To apply Corollary 4.2 we must find the values $a \in (0,1)$ for which both conditions $\widehat{\delta} < |\lambda(a)| < 1$ and $|z(a)| \leq \widehat{\gamma}$ hold. Observe that

$$|z(a)| \leq \widehat{\gamma} \;\Leftrightarrow\; |a-1+q| \geq \sqrt{pq}.$$

Hence, if $a \in (a_0, 1)$ (recall that $a_0 := 1-q-\sqrt{pq}$), then $|z(a)| > \widehat{\gamma}$. This gives (28).

Now let $a \in (0, a_0]$. Then $|z(a)| \leq \widehat{\gamma}$. Let us study $\lambda(a)$. We have $\lambda'(a) = 1 - pq/(a-1+q)^2$, so that $a \mapsto \lambda(a)$ is increasing on $(-\infty, a_0]$ from $-\infty$ to $\lambda(a_0) = r - 2\sqrt{pq}$. Thus

$$\forall a \in (0, a_0], \quad \lambda(a) \leq r - 2\sqrt{pq} < r + 2\sqrt{pq}.$$

and the equation $\lambda(a) = -(r+2\sqrt{pq})$ has a unique solution $a_1 \in (-\infty, a_0)$. Note that $a_1 < a_0$ and $\lambda(a_1) = -(r+2\sqrt{pq})$, that $\lambda(0) = p/(q-1) \in [-1, 0)$ and finally that

$$\lambda(0) - \lambda(a_1) = \frac{p}{q-1} + r + 2\sqrt{pq} = \frac{(q-\sqrt{pq}-1)^2 - 2p}{1-q}.$$

When $2p \leq (1-q+\sqrt{pq})^2$, we obtain (29). Indeed $|\lambda(a)| < r + 2\sqrt{pq}$ since

$$\forall a \in (0, a_0], \quad -(r+2\sqrt{pq}) = \lambda(a_1) \leq \lambda(0) < \lambda(a) < r + 2\sqrt{pq}.$$

When $2p > (1-q+\sqrt{pq})^2$, we have $a_1 \in (0, a_0]$ and:

- if $a \in (0, a_1)$, then (30a) holds. Indeed $r + 2\sqrt{pq} < |\lambda(a)| < 1$ since

$$\forall a \in (0, a_1], \quad -1 \leq \lambda(0) < \lambda(a) < \lambda(a_1) = -(r+2\sqrt{pq})\,;$$

- if $a \in [a_1, a_0]$, then (30b) holds. Indeed $|\lambda(a)| < r + 2\sqrt{pq}$ since

$$-(r+2\sqrt{pq}) = \lambda(a_1) \leq \lambda(a) < r + 2\sqrt{pq}.$$

□

Remark 4.2 (Discussion on the $\ell^2(\pi)$-spectral gap and the decay parameter) *As mentioned in the introduction, we are not concerned with the usual $\ell^2(\pi)$ spectral gap $\rho_2(P)$ for Birth-and-Death Markov Chains (BDMC). In particular, we can not compare our results with that of [vDS95]. To give a comprehensive discussion on [vDS95], let P be a kernel of an BDMC defined by (27) with invariant probability measure π. P is reversible with respect to π. It can be proved that the decay parameter of P, denoted by γ in [vDS95] but by γ_{DS} here to avoid confusion with our parameter γ, is also the rate of convergence $\rho_2(P)$:*

$$\gamma_{DS} = \rho_2(P) := \lim_n \|P^n - \Pi\|_2{}^{\frac{1}{n}},$$

where $\Pi f := \pi(f)\mathbf{1}$ and $\|\cdot\|_2$ denotes the operator norm on $\ell^2(\pi)$. When P is assumed to be $V_{\widehat{\gamma}}$-geometrically ergodic with $V := \{\widehat{\gamma}^n\}_{n\in\mathbb{N}}$, it follows from [Bax05, Th. 6.1], that

$$\gamma_{SD} \leq \widehat{\rho}(P).$$

Consequently the bounds of the decay parameter γ_{DS} given in [vDS95] cannot provide bounds for $\widehat{\rho}(P)$ since the converse inequality $\widehat{\rho}(P) \leq \gamma_{DS}$ is not known to the best of our knowledge. Moreover, even if the equality $\gamma_{DS} = \widehat{\rho}(P)$ was true, the bounds obtained in our Proposition 4.1 could be derived from [vDS95] only for some specific values of $P(0,0)$. Indeed the difficulty in [vDS95, p. 139-140] to cover all the values $P(0,0) \in (0,1)$ is that the spectral measure associated with Karlin and McGregor polynomials cannot be easily computed, except for some specific values of $P(0,0)$ (see [Kov09] for a recent contribution).

4.2 A non-reversible case : RWs with $g = 2$ and $d = 1$

Let $P := (P(i,j))_{(i,j)\in\mathbb{N}^2}$ be defined by

$$P(0,0) = a \in (0,1), \quad P(0,1) = 1-a, \quad P(1,0) = b \in (0,1), \quad P(1,2) = 1-b \tag{32}$$

$$\forall n \geq 2,\ P(n,n-2) = a_{-2} > 0,\ P(n,n-1) = a_{-1},\ P(n,n) = a_0,\ P(n,n+1) = a_1 > 0.$$

The form of boundary probabilities in (32) is chosen for convenience. Other (finitely many) boundary probabilities could be considered provided that P is irreducible and aperiodic. To illustrate the procedure proposed in Section 4 for this class of RWs, we also specify the numerical values

$$a_{-2} := 1/2,\ a_{-1} := 1/3,\ a_0 = 0,\ a_1 := 1/6.$$

The procedure could be developed in the same way for any other values of $(a_{-2}, a_{-1}, a_0, a_1)$ satisfying $a_{-2}, a_1 > 0$ and Condition (**NERI**) i.e. $a_1 < 2a_{-2} + a_{-1}$. Here we have

$$\phi(t) := \frac{1}{2t^2} + \frac{1}{3t} + \frac{t}{6} = 1 + \frac{1}{6t^2}(t-1)(t^2 - 5t - 3).$$

Function $\phi(\cdot)$ has a minimum over $(1, +\infty)$ at $\widehat{\gamma} \approx 2.18$, with $\widehat{\delta} := \phi(\widehat{\gamma}) \approx 0.621$. Let $V_{\widehat{\gamma}} := \{\widehat{\gamma}^n\}_{n\in\mathbb{N}}$ and let $\mathcal{B}_{\widehat{\gamma}}$ be the associated weighted space. We know from Proposition 3.2 and from irreducibility and aperiodicity properties that $\widehat{r}_{ess}(P) = \widehat{\delta}$ and P is $V_{\widehat{\gamma}}$ −geometrically ergodic (see Proposition 2.1). The polynomial $E_\lambda(\cdot)$ is

$$\forall z \in \mathbb{C}, \quad E_\lambda(z) := \frac{z^3}{6} - \lambda z^2 + \frac{z}{3} + \frac{1}{2}.$$

A simple examination of the graph of $\phi(\cdot)$ shows that $\eta = 2$. Thus the set $\mathcal{M}$ of Corollary 4.2 is $\mathcal{M} := \{(1,1),(2)\}$. Next, the constructive proof of Corollary 4.1 provides the following procedure to compute $\widehat{\rho}(P)$ (see also Remark 4.1 in the second case). Recall that $\Lambda := \{\lambda \in \mathbb{C} : \widehat{\delta} < |\lambda| < 1\}$.

First case: $\mu = (1,1)$

(a) When $\lambda \in \Lambda$ is such that $\mathcal{E}_\lambda^-$ is composed of 2 simple roots of $E_\lambda(\cdot)$, a necessary condition for λ to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ is that

$$R_1(\lambda) := \mathrm{Res}_{z_1}\big(P_1, E_\lambda\big) = 0,$$

where

$$P_1(\lambda, z_1) := \mathrm{Res}_{z_2}\big(P_0, E_\lambda\big) \quad = \quad \begin{vmatrix} 1/6 & 0 & A(\lambda,z_1) & 0 & 0 \\ -\lambda & 1/6 & B(\lambda,z_1) & A(\lambda,z_1) & 0 \\ 1/3 & -\lambda & C(\lambda,z_1) & B(\lambda,z_1) & A(\lambda,z_1) \\ 1/2 & 1/3 & 0 & C(\lambda,z_1) & B(\lambda,z_1) \\ 0 & 1/2 & 0 & 0 & C(\lambda,z_1) \end{vmatrix}.$$

and $P_0(\lambda, z_1, z_2) := A(\lambda, z_1)\, {z_2}^2 + B(\lambda, z_1)\, z_2 + C(\lambda, z_1)$ is given by

$$P_0(\lambda, z_1, z_2) := \begin{vmatrix} (1-a) & a + (1-a)z_2 - \lambda \\ (1-b)(z_1+z_2) - \lambda & b + (1-b)z_2^2 - \lambda z_2 \end{vmatrix}. \tag{33}$$

$P_0(\lambda, z_1, z_2)$ is derived using (25) from

$$P_{0,1}(\lambda, z_1, z_2) \quad := \quad \begin{vmatrix} a + (1-a)z_1 - \lambda & a + (1-a)z_2 - \lambda \\ b + (1-b)z_1^2 - \lambda z_1 & b + (1-b)z_2^2 - \lambda z_2 \end{vmatrix} = (z_1 - z_2) P_0(\lambda, z_1, z_2).$$

(b) *Sufficient part.* Consider

$$\Lambda_{(1,1)} = \mathrm{Root}\,(R_1) \cap \Lambda = \mathrm{Root}\,(R_1) \cap \big\{\lambda \in \mathbb{C} : 0.621 \approx \widehat{\delta} < |\lambda| < 1\big\}.$$

For every $\lambda \in \Lambda_{(1,1)}$:

(i) Check that $E_\lambda(z) = 0$ has two simple roots z_1 and z_2 such that $|z_i| < \widehat{\gamma} \approx 2.18$.

(ii) If (i) is OK, then test if $P_0(\lambda, z_1, z_2) = 0$ with P_0 given in (33).

If (i) and (ii) are OK, then λ is an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$.

Second case: $\mu = (2)$**.**

(a) When $\lambda \in \Lambda$ is such that $\mathcal{E}_\lambda^-$ is composed of a double root of $E_\lambda(\cdot)$, a necessary condition for λ to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ is that (see Remark 4.1)

$$Q(\lambda) = 0 \quad \text{and} \quad R_2(\lambda) := \mathrm{Res}_{z_1}\big(P_1, E_\lambda\big) = 0,$$

where

$$Q(\lambda) := \begin{vmatrix} 1/6 & 0 & 1/2 & 0 & 0 \\ -\lambda & 1/6 & -2\lambda & 1/2 & 0 \\ 1/3 & -\lambda & 1/3 & -2\lambda & 1/2 \\ 1/2 & 1/3 & 0 & 1/3 & -2\lambda \\ 0 & 1/2 & 0 & 0 & 1/3 \end{vmatrix}$$

and

$$P_1(\lambda) := \mathrm{Res}_{z_1}\big(P_0, E_\lambda\big) \quad = \quad \begin{vmatrix} 1/6 & 0 & A(\lambda) & 0 & 0 \\ -\lambda & 1/6 & B(\lambda) & A(\lambda) & 0 \\ 1/3 & -\lambda & C(\lambda) & B(\lambda) & A(\lambda) \\ 1/2 & 1/3 & 0 & C(\lambda) & B(\lambda) \\ 0 & 1/2 & 0 & 0 & C(\lambda) \end{vmatrix}.$$

where $P_0(\lambda, z_1) :=\ A(\lambda)\, z_1^2 + B(\lambda)\, z_1 + C(\lambda)$ is given by

$$P_0(\lambda, z_1) \quad := \quad \begin{vmatrix} a + (1-a)z_1 - \lambda & 1-a \\ b + (1-b)z_1^2 - \lambda z_1 & 2(1-b)z_1 - \lambda \end{vmatrix}. \tag{34}$$

(b) *Sufficient part.* Consider

$$\Lambda'_{(2)} = \mathrm{Root}\,(Q) \cap \Lambda_{(2)} = \mathrm{Root}\,(Q) \cap \mathrm{Root}\,(R_2) \cap \big\{\lambda \in \mathbb{C} : 0.621 \approx \widehat{\delta} < |\lambda| < 1\big\}.$$

For every $\lambda \in \Lambda'_{(2)}$:

(i) Check that Equation $E_\lambda(z) = 0$ has a double root z_1 such that $|z_1| < \widehat{\gamma} \approx 2.18$.

(ii) If (i) is OK, then test if $P_0(\lambda, z_1) = 0$ with P_0 given in (34).

If (i) and (ii) are OK, then λ is an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$.

Final results Define $\mathcal{Z}_{(1,1)}$ as the set of all the $\lambda \in \Lambda_{(1,1)}$ satisfying (i)-(ii) in the first case, and $\mathcal{Z}_{(2)}$ as the set of all the $\lambda \in \Lambda'_{(2)}$ satisfying (i)-(ii) in the second one. Finally set $\mathcal{Z} := \mathcal{Z}_{(1,1)} \cup \mathcal{Z}_{(2)}$. Then

$$\widehat{\rho}(P) = \max\big(\widehat{\delta}, \max\{|\lambda|,\ \lambda \in \mathcal{Z}\}\big).$$

The results (using Maple computation engine) for different instances of the values of boundary transition probabilities are reported in Table 1. In these specific examples, note that $\Lambda'_{(2)}$ is always the empty set. As expected, we obtain that $\rho_{\widehat{\gamma}}(P) \nearrow 1$ when $(a, b) \to (0, 0)$.

5 Convergence rate for RWs with unbounded increments

In this subsection, we propose two instances of RW on $\mathbb{X} := \mathbb{N}$ with unbounded increments for which estimate of the convergence rate with respect to some weighted-supremum space $\mathcal{B}_V$ can be obtained using Proposition 3.1 and Proposition 2.1. The first example is from [MS95]. The second one is a reversible transition kernel P inspired from the "infinite star" example in [Ros96]. Note that using a result of [Bax05] (see Remark 4.2), estimates of $\rho_V(P)$ with respect to $\mathcal{B}_V$ may be useful to obtain estimates on the usual spectral gap $\rho_2(P)$ with respect to Lebesgue's space $\ell^2(\pi)$. Recall that the converse is not true in general.

(a,b)	$\Lambda_{(1,1)}$	$\mathcal{Z}_{(1,1)}$	$\Lambda'_{(2)}$	$\mathcal{Z}_{(2)}$	$\widehat{\delta}$	$\widehat{\rho}(P)$
$(1/2,1/2)$	$-0.625 \pm 0.466i$, $-0.798, 0.804$	$\emptyset$	$\emptyset$	$\emptyset$	0.621	0.621
$(1/10,1/10)$	$-0.681 \pm 0.610i$ $-0.466 \pm -0.506i$ $-0.384 \pm 0.555i$	$\{-0.466 \pm 0.506i\}$	$\emptyset$	$\emptyset$	0.621	0.688
$(1/50,1/50)$	$-0.598 \pm 0.614i$ $-0.383 \pm 0.542i$ $-0.493 \pm 0.574i$ $-0.477 \pm 0.584i$ 0.994	$\{-0.493 \pm 0.574i\}$	$\emptyset$	$\emptyset$	0.621	0.757

Table 1: Convergence rate with different values of boundary transition probabilities (a,b)

5.1 A non-reversible RW with unbounded increments [MS95]

Let P be defined by

$$\forall n \geq 1,\ P(0,n) := q_n, \quad \forall n \geq 1,\ P(n,0) := p,\ P(n,n+1) := q = 1-p,$$

with $p \in (0,1)$ and $q_n \in [0,1]$ such that $\sum_{n\geq 1} q_n = 1$.

Proposition 5.1 *Assume that $\gamma \in (1, 1/q)$ is such that $\sum_{n\geq 1} q_n \gamma^n < \infty$. Then $r_{ess}(P) \leq q\gamma$. Moreover P is V_γ-geometrically ergodic with convergence rate $\rho_{V_\gamma}(P) \leq \max(q\gamma, p)$.*

Proof. We have: $\forall n \geq 1,\ (PV_\gamma)(n) = q\gamma^{n+1} + p$. Thus, if $\gamma \in (1, 1/q)$ and $\sum_{n\geq 1} q_n \gamma^n < \infty$, then Condition (**WD**) holds with V_γ, and we have $\delta_{V_\gamma}(P) \leq q\gamma$. Therefore it follows from Proposition 3.1 that $r_{ess}(P) \leq q\gamma$. Now Proposition 2.1 is applied with any $r_0 > \max(q\gamma, p)$. Let $\lambda \in \mathbb{C}$ be such that $\max(q\gamma, p) < |\lambda| \leq 1$, and let $f \in \mathcal{B}_\gamma$, $f \neq 0$, be such that $Pf = \lambda f$. We obtain $f(n) = (\lambda/q)f(n-1) - pf(0)/q$ for any $n \geq 2$, so that

$$\forall n \geq 2, \quad f(n) = \left(\frac{\lambda}{q}\right)^{n-1} \left(f(1) - \frac{pf(0)}{\lambda - q}\right) + \frac{pf(0)}{\lambda - q}.$$

Since $f \in \mathcal{B}_{V_\gamma}$ and $|\lambda|/q > \gamma$, we obtain $f(1) = pf(0)/(\lambda - q)$, and consequently: $\forall n \geq 1,\ f(n) = pf(0)/(\lambda - q)$. Next the equality $\lambda f(0) = (Pf)(0) = \sum_{n\geq 1} q_n f(n)$ gives: $\lambda f(0) = pf(0)/(\lambda - q)$ since $\sum_{n\geq 1} q_n = 1$. We have $f(0) \neq 0$ since we look for $f \neq 0$. Thus λ satisfies $\lambda^2 - q\lambda - p = 0$, that is: $\lambda = 1$ or $\lambda = -p$. The case $\lambda = -p$ has not to be considered since $|\lambda| > \max(q\gamma, p)$. If $\lambda = 1$, then $f(n) = f(0)$ for any $n \in \mathbb{N}$, so that $\lambda = 1$ is a simple eigenvalue of P on $\mathcal{B}_\gamma$ and is the only eigenvalue such that $\max(q\gamma, p) < |\lambda| \leq 1$. Then Proposition 2.1 gives the second conclusion of Proposition 5.1. □

Note that p cannot be dropped in the inequality $\rho_{V_\gamma}(P) \leq \max(q\gamma, p)$ since $\lambda = -p$ is an eigenvalue of P on $\mathcal{B}_\gamma$ with corresponding eigenvector $f_p := (1, -p, -p, \dots)$.

5.2 A reversible RW inspired from [Ros96]

Let $\{\pi_n\}_{n\in\mathbb{N}}$ be a probability distribution (with $\pi_n > 0$ for every $n \in \mathbb{N}$) and P be defined by

$$\forall n \in \mathbb{N},\ P(0,n) = \pi_n \quad \text{and} \quad \forall n \geq 1,\ P(n,0) = \pi_0,\ P(n,n) = 1 - \pi_0.$$

It is easily checked that P is reversible with respect to $\{\pi_n\}_{n\in\mathbb{N}}$, so that $\{\pi_n\}_{n\in\mathbb{N}}$ is an invariant probability distribution of P.

Proposition 5.2 *Assume that there exists $V \in [1, +\infty)^{\mathbb{N}}$ such that $V(0) = 1$, $V(n) \to +\infty$ as $n \to +\infty$ and $\pi(V) := \sum_{n\geq 0} \pi_n V(n) < \infty$. Then P is V-geometrically ergodic with $\rho_V(P) \leq 1 - \pi_0$.*

It can be checked that P is not stochastically monotone so that the estimate $\rho_V \leq 1 - \pi_0$ cannot be directly deduced from [LT96].

Proof. From $(PV)(0) = \pi(V)$ and $\forall n \geq 1,\ (PV)(n) = \pi_0 V(0) + (1-\pi_0)V(n)$, it follows that

$$PV \leq (1 - \pi_0)V + (\pi(V) + \pi_0)\, 1_{\mathbb{X}}.$$

That is, Condition (**WD**) holds true with $N := 1$, $\delta := 1 - \pi_0$ and $d := \pi(V) + \pi_0$. The inequality $r_{ess}(P) \leq 1 - \pi_0$ is deduced from Proposition 3.1.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of P and $f := \{f(n)\}_{n\in\mathbb{N}}$ be a non trivial associated eigenvector. Then

$$\lambda\, f(0) = \sum_{n=0}^{+\infty} \pi_n f(n) \qquad \text{and} \qquad \forall n \geq 1, \quad \lambda\, f(n) = \pi_0 f(0) + (1 - \pi_0) f(n). \tag{35}$$

This gives: $\forall n \geq 1, \quad f(n) = f(0)\pi_0/(\lambda - 1 + \pi_0)$. Since $f \neq 0$, it follows from the first equality in (35) that

$$\lambda = \pi_0 + \frac{\pi_0}{\lambda - 1 + \pi_0}(1 - \pi_0),$$

which is equivalent to $\lambda^2 - \lambda = 0$. Thus, $\lambda = 1$ or 0. That 1 is a simple eigenvalue is standard from the irreducibility of P. The result follows from Proposition 2.1. □

A specific instance of this model is considered in [Ros96, p. 68]. Let $\{w_n\}_{n\geq 1}$ be a sequence of positive scalars such that $\sum_{n\geq 1} w_n = 1/2$. Then P is given by

$$\forall n \in \mathbb{N}, \quad P(n,n) = 1/2 \quad \text{and} \quad \forall n \geq 1,\ P(0,n) = w_n,\ P(n,0) = 1/2$$

which is reversible with respect to its invariant probability distribution π defined by $\pi_0 := 1/2$ and $\pi_n := w_n$ for $n \geq 1$. It has been proved in [Ros96, p. 68] that, for any $X_0 \sim \alpha \in \ell^2(1/\pi)$, there exists a constant $C_{\alpha,\pi} > 0$ such that

$$\|\alpha P^n - \pi\|_{TV} \leq C_{\alpha,\pi}\ (3/4)^n \tag{36}$$

where $\|\cdot\|_{TV}$ is the total variation distance. Since we know that $\rho_2(P) \leq \rho_V(P)$ from [Bax05] and $\rho_V(P) \leq 1/2$ from Proposition 5.2, the rate of convergence in (36) is improved.

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